

Scattering of zero branes off elementary strings in Matrix theory.

Rajesh Gopakumar and Sanjaye Ramgoolam

Department of Physics, Jadwin Hall

Princeton University

Princeton, NJ 08544, USA

`rgk,ramgoola@puhep1.princeton.edu`

We consider the scattering of zero branes off an elementary string in Matrix theory or equivalently gravitons off a longitudinally wrapped membrane. The leading supergravity result is recovered by a one-loop calculation in zero brane quantum mechanics. Simple scaling arguments are used to show that there are no further corrections at higher loops, to the leading term in the large impact parameter, low velocity expansion. The mechanism for this agreement is identified in terms of properties of a recently discovered boundary conformal field theory.

1. Introduction

It is of great importance that we test the conjecture of M(atrix) theory [1] in different settings. This is specially true for the case with maximal supersymmetry compactified on low dimensional tori where the conjecture is most unambiguous. One might argue that the presence of BPS states with the right masses etc. follow from supersymmetry. Similarly the agreement between supergravity and Yang-Mills results for many of the scattering processes that have been studied is also a remnant property of the supersymmetry. In particular, the configurations have been mostly such that they reduce, in the limit when the radius of the M-direction is small, to the annulus diagram in string theory with simple D-brane boundary conditions at both ends. For this diagram it is known [2], that the leading long distance, low velocity supergravity behaviour is reproduced by the truncation to the lightest open string states.

We would therefore like to have a number of cases where agreement between Matrix theory and supergravity does not rely on this fact. One such instance is the calculation by Polchinski and Pouliot [3]. (Recently a calculation which tests a variant of the original conjecture [4] has also been performed [5].) It is our intention in this paper to carry out a one loop calculation in the quantum mechanics which does *not* correspond to an annulus diagram in a simple D-brane string theory.

The process that we are going to study is that of scattering gravitons (with M-momentum) off a longitudinal membrane in M-theory. In the string theory limit, the membrane becomes the fundamental string while the graviton becomes a 0-brane. Thus our computation may be equivalently viewed as a gauge theory calculation for 0-branes scattering off the elementary string. This involves a one-loop calculation about a non-trivial gauge field background. (Another example of such a non-trivial background was considered in [6], and rather general techniques were developed in [7].) Note also that our calculation is T-dual to the scattering of a D-string probe off a system of D-strings carrying momentum, and as such it is related to a certain limit of the scattering of D-string probe off black holes [8].

In the next section we consider the supergravity calculation at low velocities. Section 3 deals with the one loop gauge theory calculation while Section 4 considers some scaling arguments which strongly constrain the form of the one-loop answer and show that there are no corrections to our result. Section 5 discusses the mechanism for the agreement between 1-loop Yang Mills and supergravity in terms of a boundary conformal field theory. An appendix provides some details of the one loop calculation.

2. The Phase Shift in Supergravity

In this section we shall compute the leading contribution to the phase shift, in the scattering of an eleven dimensional graviton off a longitudinally wrapped membrane in supergravity. Or equivalently, when the radius R_{11} of the longitudinal M-direction becomes small, that of a zero brane off an elementary string. We also need our membrane to carry momentum in the M-direction, corresponding to the fact that in Matrix theory the object that we construct is built out of a large number of zero branes. Though, as we will see, this will not affect the leading velocity behaviour.

We write down the classical solution of such a boosted membrane in 10+1D, which was found in [9] and reviewed for example in [10]:

$$\begin{aligned} ds^2 &= H^{-2/3}[-dt^2 + dy_1^2 + dy_{11}^2 + \frac{Q_0}{r^6}(dt - dy_{11})^2] + H^{1/3}dx_i dx^i, \\ C_3 &= H^{-1}dt \wedge dy_1 \wedge dy_2, \\ H &= 1 + \frac{Q_2}{r^6} \end{aligned} \tag{2.1}$$

where $Q_2 = 8N_2(2\pi)^2(l_p)^6$ is the correctly normalised charge (see for example [11]), N_2 is an integer and $i = 2, 3 \dots 9$

The geodesic equation for a graviton in this background is:

$$|\vec{p}| \frac{dp_3}{dt} = \Gamma_{300} |\vec{p}|^2 + \Gamma_{333} (p^3)^2 + \Gamma_{3,11,11} (p^{11})^2 + 2\Gamma_{3,0,11} |\vec{p}|^2$$

with asymptotic momenta

$$p_3 = \frac{N_0 v}{R_{11}}, \quad p_{11} = \frac{N_0}{R_{11}}, \implies |\vec{p}| = \frac{N_0}{R_{11}} (1 + v^2)^{1/2}.$$

In the above, we have lowered all indices neglecting any metric dependence, justified since we will only be considering impact parameters which are large compared to Planck scale. Similarly we will only keep the leading velocity contributions.

Putting all this together gives the equation

$$\begin{aligned} |\vec{p}| \frac{dp^3}{dt} &= -\frac{Q_2 x_3}{r^8} [3(p_3)^2 (1 - \frac{Q_0}{r^6}) - 2|p_{11}|^2 \frac{Q_0}{r^6}] \\ &\quad - \frac{3Q_0 x_3}{r^8} [(|\vec{p}| - p_{11})^2 + (p_{11})^2 \frac{Q_0}{r^6}] = -\frac{3Q_2 x_3}{r^8} (p_3)^2, \end{aligned} \tag{2.2}$$

where in the last line we have neglected terms which are subleading in powers of $\frac{1}{r}$ as well as powers of velocity. We see that there is no Q_0 dependence here at all. This can

be understood since the terms proportional to Q_0 come from graviton-graviton scattering which exhibits a v^4 dependence. (The r dependence is like $\frac{1}{r^6}$ rather than the familiar $\frac{1}{r^7}$ because these are zero branes bound to the membrane and hence have one homogeneous transverse dimension.)

This corresponds to an effective force and potential:

$$\frac{\partial V}{\partial x_3} = \frac{3Q_2 x_3 v^2 N_0}{r^8 R_{11}} \implies V(r) = -\frac{Q_2 N_0 v^2}{2r^6 R_{11}}$$

which corresponds to a phase shift of

$$\delta = \frac{3\pi v Q_2 N_0}{16 R_{11} b^5} = \frac{3(2\pi)^3 v N_2 N_0 l_p^6}{4b^5 R_{11}} \quad (2.3)$$

3. One loop calculation.

The matrix quantum mechanics contains, in the large N limit, a central charge in the supersymmetry algebra corresponding to a longitudinally wrapped membrane [12]. A BPS configuration carrying this charge may be easily constructed by realising that this (winding) charge becomes momentum in a (T-dual) 1+1 dimensional Yang-Mills description.

The $U(N)$ super-quantum mechanics Lagrangian is

$$L = \frac{1}{2R_{11}} Tr \int dt \left[(D_t X^i)^2 + \frac{R_{11}^2}{4\pi^2 l_p^6} [X_i, X_j]^2 + \text{fermions} \right] \quad (3.1)$$

The Lagrangian for the 1+1 dimensional theory is:

$$L = \frac{1}{4\pi R_1 R_{11}} Tr \int dt \int_{x_1=0}^{2\pi R_1} dx_1 \left[(D_t X^i)^2 - (D_{x_1} X^i)^2 + \frac{R_{11}^2}{4\pi^2 l_p^6} [X_i, X_j]^2 + \text{fermions} \right]. \quad (3.2)$$

Here $i = 2 \dots 9$. This is obtained by the substitution

$$X_1 = \frac{1}{R_{11}} \left[\frac{i\partial}{\partial x_1} + A_1 \right]$$

in the $U(N)$ super quantum mechanics Lagrangian, which implements T-duality [1][13][14]. The variable x_1 is periodic with period $2\pi R_1$.

It will be more convenient to work with rescaled variables where

$$\begin{aligned} t_M &= t_P \frac{R_{11}}{l_p^2} (2\pi)^{-2/3} \\ X_M &= \frac{(2\pi)^{-1/3} X_P}{l_p} \\ (x_1)_M &= (x_1)_P R_{11} \\ (R_1)_M &= (R_1)_P R_{11} \end{aligned} \quad (3.3)$$

In these variables we have the Lagrangian:

$$L = \frac{1}{4\pi R_1} \text{Tr} \int dt \int_{x_1=0}^{2\pi R_1} dx_1 \left[(D_t X^i)^2 - (D_{x_1} X^i)^2 + [X_i, X_j]^2 + \text{fermions} \right], \quad (3.4)$$

There are BPS saturated states in the Yang-Mills which are essentially plane wave states, which in the original quantum mechanical picture are longitudinal membranes wrapped on the T-dual circle \tilde{X}_1 [15]. One such configuration with n units of momentum along X_1 – winding n along \tilde{X}_1 is

$$X_2 = \sqrt{\frac{2R_1}{n}} \cos \frac{n}{R_1} (x_1 + t).$$

The normalization is such that this background has the right energy.

To calculate the leading order scattering of a zero brane off this background we evaluate the path integral at one loop around the background.

$$\begin{aligned} X_2^{(1)} &= A \cos \frac{n}{R_1} (x_1 + t) \equiv \sqrt{\frac{2R_1}{n}} \cos \frac{n}{R_1} (x_1 + t) \\ X_3^{(2)} &= b \mathbf{I} \\ X_4^{(2)} &= vt \mathbf{I} \end{aligned} \quad (3.5)$$

Here we have decomposed the X 's in terms of

$$X_\mu = X_\mu^{(1)} \oplus X_\mu^{(2)}$$

where the first operator acts on the space of functions of the variable x_1 and the second operator is a $N_0 \times N_0$ matrix where N_0 is the integer charge of the probe zero brane. For simplicity we will work with $N_0 = 1$.

As usual we expand the fields around the background keeping only the quadratic terms in the fluctuations. After taking into account the ghosts and gauge-fixing terms [2][16], we separate the resulting determinants coming from the bosons, the fermions and the ghosts to the free energy,

$$W = W_B + W_G + W_F \quad (3.6)$$

In the notation of [6]

$$\begin{aligned} W_B &= \text{Tr} \log[(b^2 + P^2) \delta_{\mu\nu} + 2iF_{\mu\nu}] \\ W_G &= -2\text{Tr} \log(b^2 + P^2) \\ W_F &= -\frac{1}{2} \text{Tr} \log(b^2 + P^2 + \frac{i}{2} \Gamma_{ij} F_{ij} + \Gamma_4 v + \Gamma_1 F_{02}), \end{aligned} \quad (3.7)$$

where the non-zero elements of the matrices $F_{\mu\nu}(\mu, \nu = 0 \dots 9)$ are

$$F_{04} = -F_{40} = v; \quad F_{12} = -F_{21} = \partial_{x_1} X_2 \quad F_{02} = -F_{20} = \partial_t X_2.$$

Also

$$P^2 = \partial_t^2 - \partial_{x_1}^2 + v^2 t^2 + b^2 + A^2 \text{Cos}^2 n(x_1 + t).$$

W_B (after cancelling out the terms in W_G) can be written out as

$$\begin{aligned} W_B &= W_B^{(4)} + W_B^{(i)} \\ &= \text{Tr} \log[(b^2 + P^2) \delta_{ab} + 2iF_{ab}] \\ &+ 4 \sum_{n=0}^{\infty} \text{Tr}[\log(b^2 + P^2)], \quad (a, b = 0, 1, 2, 4), \end{aligned} \tag{3.8}$$

where the $W_B^{(4)}$ term comes from the kinetic terms for A_0, X_1, X_2, X_4 , and $W^{(i)}$ is related to the remaining six bosonic fields less two ghost degrees of freedom.

It is convenient to take a derivative:

$$\frac{dW_B}{db^2} = \frac{dW^{(4)}}{db^2} + \frac{dW^{(i)}}{db^2} \tag{3.9}$$

We will outline the calculation of the first term on the RHS above. The calculation of $W^{(i)}$ and $W^{(F)}$ is very similar.

$$\begin{aligned} \frac{dW^{(4)}}{db^2} &= \text{Tr}\left(\frac{1}{H_0 + H_1}\right) \\ &= \text{Tr}\frac{1}{H_0} - \text{Tr}\left(\frac{1}{H_0} H_1 \frac{1}{H_0}\right) + \text{Tr}\left(\frac{1}{H_0} H_1 \frac{1}{H_0} H_1 \frac{1}{H_0}\right) + \dots \end{aligned} \tag{3.10}$$

The trace includes integrals over x_1, t space as well as a sum over a discrete index running through 0, 1, 2, 4. We have separated the kinetic operator into a piece H_0 which can be diagonalized exactly, and a piece H_1 which we treat perturbatively.

$$\begin{aligned} H_0 &= (b^2 + \partial_t^2 + v^2 t^2 - \partial_{x_1}^2 + 2iF_{04}e_{04}) \\ H_1 &= A^2 \text{Cos}^2 p(x_1 + t) + 2iF_{12}e_{12} + 2iF_{02}e_{02} \end{aligned} \tag{3.11}$$

The matrix e_{ij} has non-zero entries 1 in the i 'th row and j 'th column and -1 in j 'th row and i 'th column. The dots in (3.10) represent terms which are down in the large b

expansion. Evaluation of these terms requires consideration of commutators, for example :

$$Tr\left(\frac{1}{H_0}H_1\frac{1}{H_0}\right) = Tr\left(\frac{1}{H_0^2}H_1\right) + Tr\left(\frac{1}{H_0^3}[H_0, H_1]\right) + Tr\left(\frac{1}{H_0^4}[H_0, [H_0, H_1]]\right) + \dots \quad (3.12)$$

The last term and higher are of form $\frac{1}{b^7(1+\frac{v}{b^2}+\dots)}$ and cannot contribute to the terms of interest. The second term gives vanishing contribution when evaluated.

In case of $W^{(F)}$ we again treat the $A^2 \text{Cos}^2$ term as well F_{12} and F_{02} as perturbation. Now the calculation involves an extra trace over spinor indices.

The final answer, is

$$W = \frac{3\pi v n}{2b^5}.$$

After converting to the physical variables, using (3.3), we get,

$$W = \frac{3vn(2\pi)^3 l_p^6}{4b^5 R_{11}}, \quad (3.13)$$

in agreement with (2.3) for $N_2 = n$ and $N_0 = 1$. The N_0 dependence also agrees as discussed at the end of the next section. Some more details about the calculation are given in the appendix.

4. General properties of the loop expansion

Some elementary scaling arguments can be used to derive properties of the loop expansion around backgrounds of interest in Matrix theory. These arguments have been developed in the simpler cases of flat space and non-compact orbifolds in [17][5][18]. We write down the physical Lagrangian, and then change variables to a form where the loop expansion parameter is manifest. The physical Lagrangian is:

$$L = \frac{1}{4\pi R_1 R_{11}} \int dt \int_{x_1=0}^{2\pi R_1} dx_1 [(D_t X^i)^2 - (D_{x_1} X^i)^2 + R_{11}^2 [X_i, X_j]^2 + i\Theta^T D_t \Theta - \Theta^T \Gamma^i [X_i, \Theta]] \quad (4.1)$$

We are using units where the eleven dimensional Planck length is set to 1. This is obtained from the super quantum mechanics Lagrangian by the substitution

$$X_1 = \frac{1}{R_{11}} (i \frac{\partial}{\partial x_1} + A_1). \quad (4.2)$$

We can rewrite this in terms of variables $t_h, (x_1)_h, X_h$ which are related to the above variables by:

$$\begin{aligned} t_h &= \frac{t}{R_1} \\ (x_1)_h &= \frac{x_1}{R_1} \\ X_h^i &= X^i R_1 R_{11} \\ \Theta_h &= (R_1 R_{11})^{3/2} \Theta \end{aligned} \tag{4.3}$$

In these variables the Lagrangian takes the form (numerical coefficients will not be important in this section) :

$$\frac{1}{2R_1^3 R_{11}^3} \int dt \int_{x_1=0}^{2\pi} d(x_1)_h [(D_{t_h} X_h^i)^2 - (\frac{\partial X^i}{\partial (x_1)_h})^2 + R_{11}^2 [X_i, X_j]^2 + \Theta_h^T D_t \Theta_h - \Theta_h^T \Gamma_I [X_h^I, \Theta_h]] \tag{4.4}$$

The backgrounds that we consider are therefore related by the equations:

$$\begin{aligned} b_h &= b R_1 R_{11} \\ v_h &= v R_1^2 R_{11} \end{aligned} \tag{4.5}$$

In these h-variables, where the loop expansion is manifest, the X_2 background in (3.5) is proportional to $(R_1 R_{11})^{3/2}$. The L-loop result has the form $(R_1^3 R_{11}^3)^{L-1} f_L(v_h, b_h, R_1 R_{11})$, where f_L contains positive integral powers of $(R_1 R_{11})^3$. So not all the variables b, v, R_{11}, R_1 contribute independently at a fixed loop level. In these variables the L-loop result takes the form

$$\delta_L(b, v, R_{11}, R_1) = (R_1^3 R_{11}^3)^{L-1} g_L(b_h, v_h, R_1 R_{11}) \tag{4.6}$$

At one loop this allows terms of the form

$$\begin{aligned} \frac{v}{b^5 R_{11}} &= \frac{v_h}{b_h^5} (R_1 R_{11})^3, \\ \frac{v^3 R_1}{b^5 R_{11}^2} &= \frac{v_h^3}{b_h^5} \end{aligned} \tag{4.7}$$

The first term has precisely the R_{11}, R_1 dependence of the supergravity phase shift for zero brane off elementary string. The second term has the right form for zero branes off zero branes smeared along the one dimension of the elementary string.

We can obtain further insight into the form of the phase shift computed from Yang Mills, if we organize it in the number of loops as well as the number of insertions $2K$ of the X_2 background. From (3.5), (4.5), and (4.3) it follows that $X_2^h \sim$

$(R_1 R_{11})^{3/2} \frac{1}{\sqrt{n}} \cos n((x_1)_h + t_h)$. It is significant that the only the combination $R_1 R_{11}$ appears and sits outside the cosine. Therefore loop diagrams with L loops and $2K$ insertions of the X_2 background should have the form $(R_1 R_{11})^{3(L-1+K)} f_{L,K}(v_h, b_h)$ in the h -variables

$$\delta_{L,2K}(v, b, R_1, R_{11}) = (R_1 R_{11})^{3(L-1+K)} f(v_h, b_h) \quad (4.8)$$

(We expect that the terms coming from $2K$ insertions go like n^K .) It follows from (4.8) that the $\frac{v}{R_{11} b^5}$ cannot get any corrections from generic higher loops or higher insertions. The only term that could perhaps correct it is the $L = 2, K = 0$ term but this would have no dependence on the elementary string charge, and so would be due to scattering of zero brane probe off the zero branes bound to the elementary string. But the order v term in zero-zero scattering should vanish by supersymmetry and this has been explicitly demonstrated in [19]. Therefore the $\frac{v}{b^5}$ term we are comparing with supergravity can only come in Yang Mills from a one-loop calculation with an insertion of two powers of X_2 .

Finally we comment on the range of parameters where the expansion is valid. Even at one loop there can be corrections with higher powers of $R_1 R_{11}$, $\frac{v_h}{b_h^2} = \frac{v}{b^2 R_{11}}$ and $b_h = b R_1 R_{11}$. The expansion is valid in the regime where:

$$\begin{aligned} \frac{1}{R_1 R_{11}} &\sim R_1^{(0)} \gg 1 \\ \frac{b}{R_1^{(0)}} &\gg 1 \\ \frac{v}{b^2 R_{11}} &\ll 1 \end{aligned} \quad (4.9)$$

Here $R_1^{(0)} = \frac{1}{R_1 R_{11}}$ is the physical size of the circle on which the elementary string is wrapped in M theory.

When we allow the $X^{(1)}$ block to act on an N -dimensional space tensored with functions of x_1 , and the second block to act on a N_0 dimensional space, the above discussion is modified as follows. The $N = N_0 = 1$ system considered so far corresponds to having one zero brane on a circle bound to the elementary string interacting with one zero brane probe. In general we have N zero branes bound to the elementary string, and the probe can have charge N_0 . The normalization of the perturbation gets modified by a factor $\sqrt{\frac{1}{N}}$. The one loop calculation gets a factor NN_0 from the multiplicity of off-diagonal fields. So the leading answer is modified by a factor proportional to N_0 , as expected from supergravity. In general the 1-loop answers with $2K$ insertions pick up a factor of N^K . At higher loops the leading large N behaviour is N^{L-1+K} .

5. String Diagrams and the Yang Mills calculation

Let us review a few features of the simpler one-loop calculations which have been used to test of Matrix Theory. The simplest case of scattering of zero branes off zero branes relies on the agreement between the short and long distance behaviours of closed string exchange amplitudes (cylinder diagram) [2]. We would like to ask what kind of property of string amplitudes is responsible for the agreement between the 1-loop Yang Mills result and the scattering of zero branes off an elementary string source found in supergravity. Conversely, we can obtain information about the string theory formulation of the problem of scattering zero branes off an elementary string background, by analysing the Yang Mills Matrix theory calculation.

The system of zero-brane with elementary string is T-dual to D-string carrying momentum. This system can be described by adding to the flat space closed string worldsheet action boundary operator [20] of the form:

$$|B\rangle = e^{\mathcal{O}}|B_0\rangle \quad (5.1)$$

Here $|B_0\rangle$ is the boundary operator for a D-string alone. The exponential is responsible for turning on momentum. We can expand the exponential and think of B as creating a boundary with D-string boundary conditions and the powers of \mathcal{O} as insertions at the boundary. In our case the relevant form of \mathcal{O} has been considered by Callan and Klebanov in [20], who showed that it defines a valid conformal field theory:

$$\mathcal{O} = \int d\sigma a_p e^{ipX^+} \left(\epsilon^j \frac{\partial}{\partial \tau} X^j + ip\psi^+ \psi^j \right) + c.c. \quad (5.2)$$

In the Yang Mills calculation, the diagram that leads to the leading effect is of the form shown in Fig.1.

We have drawn a composite operator F^2 because the contributing terms involves the perturbation \bar{X}_2 in the form $\int dx (\partial_t \bar{X}_2(x))^2$. This clearly corresponds to a degeneration of the string theory diagram with two insertions of the \mathcal{O} operator at the boundary, as shown in fig. 2.

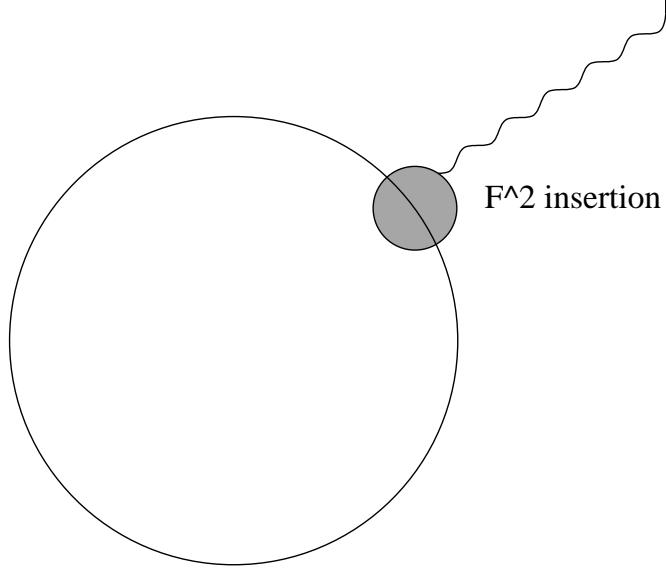


Fig. 1: The one loop diagram with an insertion

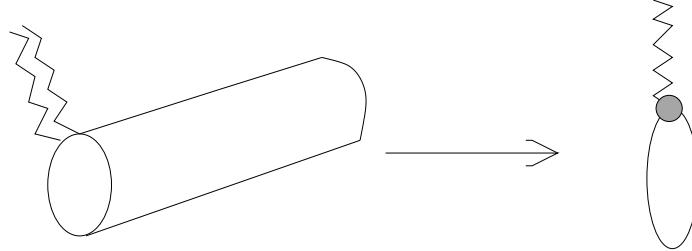


Fig. 2: Degeneration of the zero brane string scattering diagram

So the mechanism for the agreement between Matrix theory and supergravity is again the fact that for this cylinder diagram the contributions of the long multiplets cancel, so that it suffices to look at the lightest open string modes [21][22][23][24]. It will be very interesting to understand in some generality what class of boundary CFTs allows such a decoupling. In particular it will be interesting to see if they include those describing systems of D-branes, which are relevant in black hole physics [25][26]. This should shed further light on the Matrix theory approaches to black holes that have been started in [8][27][28][29][30].

To describe just the elementary string rather than elementary string with zero branes, (or in the T -dual version plane wave without D-string) a guess is that we just replace $|B_0\rangle$ in the boundary operator (5.2) by the ordinary vacuum of the bulk CFT, rather than the D-string boundary state.

6. Discussion and conclusion

We have studied scattering off elementary strings bound to zero branes in Matrix theory. We did a one loop calculation in the superquantum mechanics to get the leading long distance, low velocity term in the phase shift. We argued, using scaling and supersymmetry, that the loop expansion does not correct this leading phase shift.

A point to be noted is that in the supergravity calculation we used a simple solution which is expected [31] to have the correct long distance properties of a longitudinal membrane carrying momentum. A choice of profile may be necessary for precise matching at higher orders in the long distance expansion.

Systems with momentum, D1-brane and D5-brane charge are of direct interest in the Matrix theory description of black holes. We have seen in this paper that backgrounds with somewhat non-trivial time dependence are correctly incorporated in Matrix theory. As in [17][5][18] we have seen here that simple dimensional arguments give a lot of information about the phase shifts. We expect this to be quite general. Further it will be interesting to explore the detailed connection between Matrix Superquantum mechanics and degenerations of diagrams in open-closed string theories.

Several other interesting questions can be addressed in these models. It is important to know how far we can go in recovering the long distance expansion of scattering amplitudes without knowing the detailed form of the zero brane bound state wavefunctions. This has been discussed to some extent in [17][5]. It is also important to understand possible relations of the Matrix loop expansion to the non-compact limit of M-theory [32].

Acknowledgments

It is a pleasure to thank V. Balasubramanian, O. Ganor, Z. Guralnik, A. Hashimoto, F. Larsen, V. Periwal, L. Susskind, O. Tafjord and A. Tseytlin for helpful conversations. This work was supported by NSF Grant PHY96-00258 and DOE Grant DE-FGO2-91-ER40671.

Appendix

Here we put together some details of the one loop computation. The explicit evaluation of the determinants is done in Euclidean space by the substitution $t \rightarrow it$, $v \rightarrow -iv$. For instance, in calculating $W^{(4)}$ using (3.10) and (3.11) we had to deal with $Tr \frac{1}{H_0}$, $Tr(\frac{1}{H_0^2} H_1)$, $Tr(\frac{1}{H_0^3} [H_0, H_1])$ and $Tr(\frac{1}{H_0^3} H_1^2)$.

Thus, diagonalising the 4×4 matrix in H_0 , gives

$$\begin{aligned} Tr \frac{1}{H_0} = & 2tr \frac{1}{b^2 + \partial_t^2 + v^2 t^2 - \partial_{x_1}^2} \\ & + tr \frac{1}{b^2 + \partial_t^2 + v^2 t^2 - \partial_{x_1}^2 + 2v} \\ & + tr \frac{1}{b^2 + \partial_t^2 + v^2 t^2 - \partial_{x_1}^2 - 2v} \end{aligned} \quad (6.1)$$

where the trace tr is only over x_1, t space. Each of these traces can be evaluated and written in the form

$$\sum_{n=0}^{\infty} \int dp \frac{1}{p^2 + b^2 + iv(2n+a)} \quad (6.2)$$

with $a = \pm 1, 3$.

The integral and sum can then be performed by using

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(b^2 + iv(2n+a))^{1/2}} = & \frac{1}{\pi^{1/2} b} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dx e^{-[1+(2n+a)\gamma]x^2} \\ = & \frac{1}{\pi^{1/2} b} \int_{-\infty}^{\infty} dx e^{-x^2} \frac{e^{-(a-1)\gamma x^2}}{2 \sinh(\gamma x^2)}. \end{aligned} \quad (6.3)$$

Here $\gamma = \frac{iv}{b^2}$.

Similarly, in evaluating $Tr(\frac{1}{H_0^2} H_1)$ one first diagonalises the 4×4 matrix in H_0 and conjugates H_1 by the appropriate unitary matrix. One obtains

$$\begin{aligned} Tr(\frac{1}{H_0^2} H_1) = & 2tr \frac{1}{(b^2 + \partial_t^2 + v^2 t^2 - \partial_{x_1}^2)^2} A^2 \cos^2 \frac{n}{R_1} (x_1 + t) \\ & + tr \frac{1}{(b^2 + \partial_t^2 + v^2 t^2 - \partial_{x_1}^2 + 2v)^2} A^2 \cos^2 \frac{n}{R_1} (x_1 + t) \\ & + tr \frac{1}{(b^2 + \partial_t^2 + v^2 t^2 - \partial_{x_1}^2 - 2v)^2} A^2 \cos^2 \frac{n}{R_1} (x_1 + t). \end{aligned} \quad (6.4)$$

The trace is carried out similarly. There is an average over the \cos^2 term as well. There are sums of the form $\sum_{n=0}^{\infty} \frac{1}{(b^2 + iv(2n+a))^{3/2}}$, which are evaluated by differentiating (6.3) with respect to b^2 . The term $Tr(\frac{1}{H_0^3} [H_0, H_1])$ vanishes because $(\partial_{x_1}^2 - \partial_t^2) A^2 \cos^2 p(x_1 + t)$ vanishes and because $\partial A^2 \cos^2 p(x_1 + t)$ gives zero when integrated over x .

The crucial contribution comes from the term $Tr(\frac{1}{H_0^3}H_1^2)$. Diagonalising as before gives us

$$\begin{aligned}
Tr(\frac{1}{H_0^3}H_1^2) = & tr \frac{1}{(b^2 + \partial_t^2 + v^2 t^2 - \partial_{x_1}^2)^3} (2A^4 \text{Cos}^4 \frac{n}{R_1} (x_1 - t) - 4(F_{02}^2 + 2F_{12}^2)) \\
& + tr \frac{1}{(b^2 + \partial_t^2 + v^2 t^2 - \partial_{x_1}^2 + 2v)^3} (A^4 \text{Cos}^4 \frac{n}{R_1} (x_1 - t) - 2F_{02}^2) \\
& + tr \frac{1}{(b^2 + \partial_t^2 + v^2 t^2 - \partial_{x_1}^2 - 2v)^3} (A^4 \text{Cos}^4 \frac{n}{R_1} (x_1 - t) - 2F_{02}^2)
\end{aligned} \tag{6.5}$$

Finally adding these up gives

$$\begin{aligned}
\frac{dW^{(4)}}{db^2} = & \frac{1}{2\pi^{1/2}b} \int_{-\infty}^{\infty} dx e^{-x^2} \frac{[1 + \cosh(2\gamma x^2)]}{2\sinh(\gamma x^2)} \\
& + \frac{A^2}{4\pi^{1/2}b^3} \int_{-\infty}^{\infty} dx x^2 e^{-x^2} \frac{[1 + \cosh(2\gamma x^2)]}{\sinh(\gamma x^2)} \\
& + \frac{3A^4}{32\sqrt{\pi}b^5} \int dx \frac{x^4 e^{-x^2}}{\sinh(\gamma x^2)} [1 + \text{Cosh}(2\gamma x^2)] \\
& - \frac{\langle F_{12}^2 \rangle}{2\pi^{1/2}b^5} \int_{-\infty}^{\infty} dx x^4 e^{-x^2} \frac{[1 - \cosh(2\gamma x^2)]}{\sinh(\gamma x^2)}
\end{aligned} \tag{6.6}$$

where $\langle F_{12}^2 \rangle = \int_0^{2\pi R_1} dx_1 F_{12}^2$.

The calculation for $W^{(i)}$ can be performed in a similar manner. We obtain

$$\begin{aligned}
\frac{dW^{(i)}}{db^2} = & \frac{1}{\pi^{1/2}b} \int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{\sinh(\gamma x^2)} \\
& + \frac{A^2}{2\pi^{1/2}b^3} \int_{-\infty}^{\infty} dx x^2 \frac{e^{-x^2}}{\sinh(\gamma x^2)} \\
& + \frac{3A^4}{16\pi^{1/2}b^5} \int_{-\infty}^{\infty} dx x^4 \frac{e^{-x^2}}{\sinh(\gamma x^2)}.
\end{aligned} \tag{6.7}$$

Then there is the fermion contribution which is evaluated in a way very similar to that of $W^{(4)}$. Now the trace includes one over the $16 \times 16 \Gamma$ matrices. Most of the terms vanish since the trace of a product of one, two and three distinct Γ matrices is zero. This

gives as a result

$$\begin{aligned}
\frac{dW_F}{db^2} = & -\frac{2}{\pi^{1/2}b} \int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{\sinh(\gamma x^2)} \cosh(\gamma x^2) \\
& -\frac{A^2}{\pi^{1/2}b^3} \int dx x^2 \frac{e^{-x^2}}{\sinh(\gamma x^2)} \cosh(\gamma x^2) \\
& -\frac{3A^4}{8\pi^{1/2}b^5} \int_{-\infty}^{\infty} dx x^4 \frac{e^{-x^2}}{\sinh(\gamma x^2)} \cosh(\gamma x^2) \\
& +2 \frac{\langle (F_{12}^2 - F_{02}^2) \rangle}{\pi^{1/2}b^5} \int_{-\infty}^{\infty} dx x^4 \frac{e^{-x^2}}{\sinh(\gamma x^2)} \cosh(\gamma x^2).
\end{aligned} \tag{6.8}$$

In the sum $\frac{dW^{(4)}}{db^2} + \frac{dW^{(i)}}{db^2} + \frac{dW^{(F)}}{db^2}$, the only non-vanishing contribution to order $\frac{v}{b^7}$ is seen to come from the F_{12}^2 term in (6.6).

$$\frac{dW}{db^2} = \frac{15\pi v}{4b^7} \tag{6.9}$$

The rest of the terms start at order $\frac{v^3}{b^7}$.

References

- [1] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, “*M Theory As A Matrix Model: A Conjecture*,” *hep-th/9610043*
- [2] M.R. Douglas, D. Kabat, P. Pouliot and S. Shenker, “*D-branes and Short Distances in String Theory*,” *hep-th/9608024*
- [3] J. Polchinski, P. Pouliot, “*Membrane Scattering with M-Momentum Transfer*,” *hep-th/9704029*
- [4] L. Susskind, “*Another Conjecture about M(atrix) Theory*,” *hep-th/9704080*
- [5] M. Becker, K. Becker, J. Polchinski, A. Tseytlin “*Higher Order Graviton Scattering in M(atrix) Theory*” *hep-th/9706072*
- [6] I. Chepelev and A. Tseytlin, “*Long-distance interactions of D-brane bound states and longitudinal 5-brane in M(atrix) theory*” *hep-th/9701151*
- [7] I. Chepelev and A. Tseytlin, “*Interactions of type IIB D-branes from D-instanton matrix model*,” *hep-th/9705120*.
- [8] M. Douglas, J. Polchinski, A. Strominger, “*Probing Five-Dimensional Black Holes with D-branes*,” *hep-th/9703031*
- [9] A. Tseytlin, “*Harmonic superpositions of M-branes*,” *NPB475* (1996) 149, *hep-th/9604035*
- [10] J.G. Russo, “*BPS Bound States, Supermembranes and T-Duality in M-Theory*,” *hep-th/9703118*
- [11] V. Balasubramanian and F. Larsen, “*Relativistic Brane Scattering*,” *hep-th/9703039*
- [12] T. Banks, N. Seiberg, S. Shenker, “*Branes from Matrices*”, *hep-th/9612157*
- [13] W. Taylor, “*D-brane field theory on compact spaces*,” *hep-th/9611042*
- [14] O. Ganor, S. Ramgoolam, W. Taylor, “*Branes, Fluxes and Duality in M(atrix)-Theory*”, *hep-th/9611202*
- [15] R. Gopakumar, “*BPS States In Matrix Strings*”, *hep-th/9704030*
- [16] G. Lifschytz and S. Mathur, “*Supersymmetry and Membrane Interactions in M(atrix) Theory*,” *hep-th/9612087*
- [17] O. Ganor, R. Gopakumar, S. Ramgoolam, “*Higher loop effects in M(atrix) Orbifolds*”, *hep-th/9706072*
- [18] L. Susskind as referenced in [5]
- [19] K. Becker and M. Becker, “*A Two-loop test of M(atrix)-theory*”, *hep-th/9705091*
- [20] C. Callan, I. Klebanov, “*D-brane boundary state dynamics*” *hep-th/9511173*
- [21] M. Douglas and M. Li, “*D-Brane Realization of N=2 Super Yang-Mills Theory in Four Dimensions*, *hep-th/9604041*
- [22] M. Green and M. Gutperle “*Light-cone supersymmetry and D-branes*” *hep-th/9604091*
- [23] J. Harvey and G. Moore, “*Algebras, BPS States, and Strings*” *hep-th/9510182*
- [24] C. Bachas and E. Kiritsis, “*F⁴ Terms in N = 4 String Vacua*” *hep-th/9611205*

- [25] A. Strominger and C. Vafa “*Microscopic Origin of the Bekenstein-Hawking Entropy*,” *hep-th/9601029*
- [26] C. Callan and J. Maldacena, “*D-brane Approach to Black Hole Quantum Mechanics*” *hep-th/9602043*
- [27] M. Li and E. Martinec “*Matrix Black Holes*” *hep-th/9703211*
- [28] R. Dijkgraaf, H. Verlinde, E. Verlinde, “*5D Black Holes and Matrix Strings*,” *hep-th/9704018*
- [29] M. Li and E. Martinec, “*On the Entropy of Matrix Black Holes*,” *hep-th/9704134*
- [30] E. Halyo, “*M(atrix) black holes in five dimensions*,” *hep-th/9705167*.
- [31] J.G. Russo, and A.A. Tseytlin, “*Waves, boosted branes and BPS states in M-theory*,” *hep-th/9611047*
- [32] L. Susskind, talk at Strings 97.